Self-consistent expansion results for the nonlocal Kardar-Parisi-Zhang equation

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In this paper various predictions for the scaling exponents of the nonlocal Kardar-Parisi-Zhang (NKPZ) equation are discussed. I use the self-consistent expansion (SCE), and obtain results that are quite different from the result obtained in the past, using dynamic renormalization-group analysis, a scaling approach, and a self-consistent mode-coupling approach. It is shown that the results obtained using SCE recover an exact result for a subfamily of the NKPZ models in one dimension, while all the other methods fail to do so. It is also shown that the SCE result is the only one that is compatible with simple observations on the dependence of the dynamic exponent z in the NKPZ model on the exponent ρ characterizing the decay of the nonlinear interaction. The reasons for the failure of other methods to deal with NKPZ are also discussed.

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I. INTRODUCTION

The field of disorderly surface growth has received much attention during the last two decades. A multitude of different phenomena such as fluid flow in porous media, propagation of flame fronts, flux lines in superconductors not to mention deposition processes, bacterial growth, and "DNA walk" [1] are all said to be related to the famous Kardar-Parisi-Zhang (KPZ) equation [2]. Therefore, further understanding of the behavior of the KPZ equation has a broad interest in the fields of nonequilibrium dynamics and in disordered systems.

The KPZ equation for a growing surface is

$$\frac{\partial h(\vec{r},t)}{\partial t}=\nu\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2+\eta(\vec{r},t). \tag{1}$$

This equation describes the height fluctuations of a d-dimensional interface, where $h(\vec{r},t)$ is the height at $\vec{r}$ measured relative to its spatial average, ν is the diffusion constant, and $\eta(\vec{r},t)$ is the fluctuation of the rate of deposition. The noise $\eta(\vec{r},t)$ has a zero mean $[\langle \eta(\vec{r},t)\rangle=0]$ and it satisfies

$$\langle \eta(\vec{r},t)\,\eta(\vec{r}\,',t')\rangle=2D_0\,\delta^d(\vec{r}-\vec{r}\,')\,\delta(t-t') \tag{2}$$

(white noise) where d is the substrate dimension and D_0 specifies the noise amplitude.

Together with the great success of the KPZ equation to describe many growth models and phenomena, there has been a growing pool of data that is not well described by KPZ, and triggered further research. One of the first classes that belongs to this non-KPZ behavior is the well-known molecular-beam-epitaxy (MBE) class (sometimes also called the conserved KPZ equation) [3–9]. This class is distinct from KPZ in having surface diffusion as the basic relaxation mechanism. However, the modified behavior introduced by the MBE equation is not at all sufficient to account for all the rich non-KPZ experimental data in the field. Actually, the conserved KPZ model is not considered today as a realistic continuum model for MBE processes. The conserved KPZ model does not include the effects of step-edge barriers and the phenomenon of slope selection, which are experimentally well known to dominate the MBE growth for long times (see Refs. [10,11]), so that the inclusion of such mechanisms enriches the possible universality classes even more.

A different line of research suggested that the basic growth equation should not be changed. Instead, the white noise that appears in the original KPZ equation should be correlated—either temporally or spatially [12–16]. This approach was indeed quite successful. However, it still failed to give a good account for all the measured scaling exponents.

Recently, some researchers suggested that incorporating the long-range nature of interactions in the growing surface is necessary for a proper description of many systems such as colloid systems [17], paper-burning experiments [18], or protein deposition kinetics [19]. Following this basic intuition Mukherji and Bhattacharjee [20] developed a Langevin-type equation with a nonlocal nonlinearity, thus going beyond the KPZ local description of interactions. They studied originally the white noise case that was later generalized to spatially correlated noise by Chattapohadhyay [21]. To be more specific, the equation they studied was

$$\frac{\partial h(\vec{r},t)}{\partial t}=\nu\nabla^2 h(\vec{r},t)+\frac{1}{2}\int dr'\,g(\vec{r}\,')\,\nabla h(\vec{r}+\vec{r}\,',t)\times\nabla h(\vec{r}-\vec{r}\,',t)+\eta(\vec{r},t), \tag{3}$$

where the kernel $g(\vec{r})$ represents the long-range interactions. They take $g(\vec{r})$ with a short-range part $\lambda_0\delta^d(\vec{r})$ and with a long-range part $\sim\lambda_\rho r^{\rho-d}$, or more precisely in Fourier space, $\hat{g}(q)=\lambda_0+\lambda_\rho q^{-\rho}$. The noise has again a zero mean, but it is allowed to have spatially long-range correlations, characterized by its second moment

$$\langle \eta(\vec{r},t)\,\eta(\vec{r}\,',t)\rangle=2D_0|\vec{r}-\vec{r}\,'|^{2\sigma-d}\delta(t-t'), \tag{4}$$

where the case of white noise is restored in the limit $\sigma=0$ [20].

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In the following section I will present current theoretical predictions for the scaling exponents of the NKPZ model. It will become evident that all the previous results are inconsistent with an exact one-dimensional result obtained in the past. Then, in Sec. III the self-consistent expansion (SCE) approach will be applied to this model. Interestingly, this method yields results that are consistent with the exact one-dimensional result. Then, results for higher dimensions as well as for general ρ's are presented. Eventually, a discussion of the reasons for the failure of the various theoretical methods is given in Sec. IV, and conclusions are drawn in Sec. V.

II. PREVIOUS RESULTS

Both papers [20,21] investigated this problem using dynamic renormalization-group (DRG) analysis, and derived a very complex zoo of universality classes (Ref. [21] also applied a mode-coupling approach confirming some of the results of DRG). Let us not get into all those details, but rather focus on the strong-coupling solution (in the KPZ sense [1]) suggested by both papers, namely,

$$z_{DRG}=2+\frac{(d-2-2\rho)(d-2-3\rho)}{(3+2^{-\rho})d-6-9\rho}, \tag{5}$$

where z is the dynamic exponent. To complete the picture, the roughness exponent α can be calculated using the modified scaling relation $\alpha+z=2-\rho$ (which actually comes from the famous Galilean invariance). Notice that this result does not depend explicitly on σ as long as $\sigma>0$.

On the basis of the DRG result [Eq. (5)] Mukherji and Bhattacharjee tried to explain some non-KPZ experimental results in one dimension, namely, a roughness exponent of $\alpha=0.71$ given in Refs. [17,18] that implies, according to their DRG calculation, $\rho=-0.12$, and requires $\lambda_0=0$. As already appreciated by the authors, this result raises some doubts on physical grounds, whether a true $\lambda_0=0$ is seen in experiment and where should a negative ρ, which implies anticorrelations [22] rather than correlations that cause the interaction, come from.

This finding is obviously disturbing because if DRG yields accurate results then the NKPZ equation cannot be relevant to the physical processes discussed above. As will be shown later, the DRG result is far from being accurate, so that the above argument against the relevance of the NKPZ equation is not valid. The relevance of the NKPZ equation has been doubted, however, for other reasons too, but this will not be discussed here.

The equation is interesting by itself and generalizes the traditional KPZ prototype of nonlinear stochastic field equations that are so abundant in the description of natural phenomena. Therefore, reliable methods of solution are of great importance. DRG is one such general method, but unfortunately the DRG results look suspicious for the following three reasons.

First, the expression for the dynamic exponent [Eq. (5)] reduces to the well-known result of Medina *et al.* [12] $z=2+(d-2)^2/(4d-6)$ in the limit of $\rho=0$ (i.e., the limit of local KPZ)—a result that by itself is meaningful only for the case of $d=1$. It is a well-known fact that DRG cannot give the strong-coupling solution of the local KPZ equation for $d>1$ [24]. This may suggest that the DRG result is not correct also for the nonlocal case, especially for $d>1$. Furthermore, the mode-coupling approach that is known to be more appropriate than DRG when dealing with the standard KPZ problem [13,25] yielded a strong-coupling result that is different from Eq. (5) [21]. Yet, it should be mentioned that the mode-coupling approach [21] (at least using the simplest ansatz) was not able to produce a nonpower counting solution, as obtained in the local KPZ case [25].

Second, it is easily seen that the explicit expression for the dynamical exponent [given in Eq. (5) above] has a singularity for ρ's that satisfy $(3+2^{-\rho_s})d-6-9\rho_s=0$ (for example, in one dimension $\rho_s=-0.205$). It is not clear why should such a singularity appear at all and if it appears, then why should it take this specific value.

Third, from general considerations one should expect the following scenario: as ρ becomes more and more negative, the interaction that couples the gradients is enormously reduced with distance [remember that the long-range part of the kernel scales as $g(\vec{r})\sim\lambda_\rho r^{\rho-d}$]. Therefore, one should expect that in this region, the nonlinear term becomes irrelevant, and the scaling exponents of the linear theory (also known as the Edwards-Wilkinson (EW) theory [23]) should be recovered. On the other side, as ρ becomes larger and larger, the interaction becomes more and more relevant, and one might expect that as a consequence faster relaxations should appear in the system (faster relaxations imply smaller values of the dynamical exponent z). Therefore, it is reasonable that a nonincreasing behavior of the dynamical exponent z as a function of ρ will be seen. It is not difficult to see that the expression for the dynamical exponent [given by Eq. (5)] does not follow this reasoning.

Lately, an exact result for the NKPZ model was found [26]. It turns out that the Fokker-Planck equation associated with the Langevin-form of the NKPZ model [i.e., Eq. (3) above] can be solved exactly for a specific subfamily of models in one dimension. More specifically, a Gaussian steady-state solution was found for the case $\lambda_0=0$, $\rho=2\sigma$ when $d=1$. This exact solution yields the following dynamic exponent:

$$z_{exact}=\frac{3-3\rho}{2}. \tag{6}$$

It is easy to see that this result reduces to the well-known local-KPZ (that corresponds to $\rho=0$) result, $z=3/2$, in one dimension.

This exact result is not compatible with the DRG result—either quantitatively or qualitatively. As mentioned above, DRG is usually considered relatively reliable in one dimension. However, this exact result indicates DRG's shortcoming already in one dimension. The inevitable conclusion is that in order to gain insight into the behavior of the NKPZ model in any dimension, and for any value of ρ (the nonlocal parameter) and σ (the long-range noise parameter) it is necessary to employ more reliable methods.

Actually, two methods that proved useful in the context of the local-KPZ problem were applied recently to the NKPZ problem as well. The first method, is a Flory-type scaling approach that was originally proposed by Hentschel and Family [27] and generalized lately by Tang and Ma [28] to the nonlocal case. The strong-coupling dynamical exponent obtained by using this scaling approach (SA) is

$$z_{SA}=\frac{(2-\rho)(2+d-2\sigma)}{3+d-2\sigma}. \tag{7}$$

The second method applied to the NKPZ equation is an improved version of the mode-coupling approach. More precisely, it is a solution of the mode-coupling equations using an improved ansatz that describes correctly the asymptotic behavior of the solution. This approach was first proposed by Colaiori and Moore [29], and applied by Hu and Tang [30] to the nonlocal case. The strong-coupling exponents using this method are obtained by a numerical solution of the following set of transcendental equations:

$$\frac{PAS_d'}{B}=\frac{d(2-z)}{\pi z^2}\,\frac{2^{2\rho}}{BI(B,z,\rho)},$$

$$\frac{PAS_d'}{B^2}=\frac{\beta(\beta-z)}{\pi z^2}\,\frac{2^{(2z-\beta)/\beta+2\rho}}{\Gamma\left(\dfrac{2z-\beta}{\beta}\right)/\beta}, \tag{8}$$

where $S_d'=\int_0^\pi \sin^{d-2}\theta d\theta$, $\Gamma(u)$ is Euler's gamma function, $\beta=d+4-2z-2\rho$, $B=1/2(2-z)$, and $I(B,z,\rho)=\int_0^\infty(1-\rho-2s^2)s^{2z-3}\exp(-B^{\beta/z}s^\beta-s^2)ds$. Such a numerical solution for z as a function of ρ for $d=1,2,3$ is presented in Fig. 1 of Ref. [30].

While these two results look reasonable, in the sense that both do not develop any singularity as a function of ρ, and at least the first one [given by Eq. (7)] describes a nonincreasing dynamical exponent, it turns out that both results are still incompatible with the exact result of $z=(3-3\rho)/2$ [given by Eq. (6)]. In addition, the mode-coupling solution has a peak for $d>1$ meaning that the dynamic exponent z is not monotonous as a function of ρ, which is expected from general considerations. Therefore, there is still a need for a reliable method to tackle this problem. The reasons given above motivated the analysis presented in this paper.

III. THE SELF-CONSISTENT EXPANSION

In this paper I apply a method developed by Schwartz and Edwards [16,31,32] (also known as the self-consistent expansion approach). This method has been previously applied successfully to the KPZ equation. The method gained much credit by being able to give a sensible prediction for the KPZ critical exponents in the strong-coupling phase for $d>1$, where, as previously mentioned, many renormalization-group (RG) approaches failed (as well as DRG of course). It also produces the exact one-dimensional result. I will show that in contrast to DRG, the experimental result $\alpha=0.71$ is accounted for, when using the SCE, by a positive ρ. Furthermore, I will show that the SCE method yields the exact result, for the specific subfamily of models that can be solved exactly in one dimension, where other methods fail. In addition, solutions for any d, ρ, and σ are obtained. These solutions are consistent with the expected qualitative scenario presented above.

As will be seen immediately, another remarkable advantage of the SCE method is the minor changes needed in order to generalize the result of local KPZ with uncorrelated noise to include nonlocal interactions as well as spatially correlations in the noise. The above implies a second important motivation for this paper, namely, a demonstration of the robustness of the SCE method as well as its mathematical coherence and consistency.

The SCE method is based on going over from the Fourier transform of the equation in Langevin form to a Fokker-Planck form and constructing a self-consistent expansion of the distribution of the field concerned. I therefore consider first the NKPZ equation [Eq. (3)] in Fourier components

$$\frac{\partial h_q}{\partial t}=-\nu_q h_q-\sum_{\ell,m} M_{q\ell m}h_\ell h_m+\eta_q, \tag{9}$$

where Ω is the volume of the system, to be taken eventually to infinity and η_q is the noise term that satisfies

$$\langle \eta_q(t)\rangle=0, \tag{10}$$

$$\langle \eta_q(t)\eta_{q'}(t')\rangle=2D_0 q^{-2\sigma}\delta^d(\vec{q}+\vec{q}\,')\delta(t-t'), \tag{11}$$

and $M_{q\ell m}$ and ν_q are defined by

$$M_{q\ell m}=\frac{\lambda_\rho}{\sqrt{\Omega}}q^{-\rho}(\vec{\ell}\cdot\vec{m})\delta_{q,\ell+m},$$

$$\nu_q=\nu q^2. \tag{12}$$

Rewriting this equation in a Fokker-Planck form we get

$$\frac{\partial P}{\partial t}+\sum_q \frac{\partial}{\partial h_q}\left[D_0 q^{-2\sigma}\frac{\partial}{\partial h_{-q}}+\nu_q h_q+\sum_{\ell,m} M_{q\ell m}h_\ell h_m\right]P=0, \tag{13}$$

where $P\{h_q,t\}$ is the probability of having h_q at a specific time t.

The expansion is formulated in terms of ϕ_q and ω_q, where ϕ_q is the two-point function in momentum space, defined by $\phi_q=\langle h_q h_{-q}\rangle_S$ (the subscript S denotes steady state-averaging) and ω_q is the characteristic frequency associated with each mode.

It is generally expected that for small enough q, ϕ_q and ω_q are power laws in q:

$$\phi_q=Aq^{-\Gamma} \tag{14}$$

and

$$\omega_q=Bq^z, \tag{15}$$

TABLE I. A complete description of all the possible phases of the NKPZ problem, for any value of d,ρ, and σ. The first two columns give the scaling exponents z and Γ for a particular phase, and the third column states each phase's validity condition. Note that $\Gamma_0(d,\rho)$ is the numerical solution of the transcendental equation $F(\Gamma,z,\rho)=0$ with the scaling relation $z=(d+4-\Gamma-2\rho)/2$.

z	Γ	Validity
$2-2\rho$	$2-2\rho+2\sigma$	$\sigma>\rho>0$ and $d>2+2\sigma-3\rho$
$2-2\rho$	2	$\rho>0$, $\sigma<\rho$, and $d>2-\rho$
2	$2+2\rho$	$\sigma<\rho<0$ and $d>2+4\rho$
2	$2+2\sigma$	$\rho<0$, $\sigma>\rho$, and one of the following: (1) $d>\max\{2+2\sigma+\rho,2+4\sigma\}$ (2) $\sigma>\frac{1}{2}\rho$ and $d>2+2\sigma+2\rho$ (3) $\sigma<\frac{1}{2}\rho$ and $2+4\sigma<d<2+2\sigma+\rho$ (4) $2+2\sigma+2\rho<d<2+4\sigma$
$\frac{d+4-2\rho-2\sigma}{3}$	$\frac{d+4-2\rho+4\sigma}{3}$	$\sigma>\rho$, $d<2+2\sigma-\rho-3\|\rho\|$, and $d+4-2\rho+4\sigma>3\Gamma_0(d,\rho)$
$\frac{d+4-\Gamma_0(d,\rho)-2\rho}{2}$	Sol. of $F(\Gamma,z,\rho)=0$, denoted by $\Gamma_0(d,\rho)$	$d<3\Gamma_0(d,\rho)-4+\min\{2\rho-4\sigma,-2\rho\}$ and $d<\Gamma_0(d,\rho)-2\|\rho\|$

where z is just the dynamic exponent, and the exponent Γ is related to the roughness exponent α by

$$\alpha=\frac{\Gamma-d}{2}. \tag{16}$$

The main idea is to write the Fokker-Planck equation $\partial P/\partial t=OP$ in the form $\partial P/\partial t=[O_0+O_1+O_2]P$, where O_0 is to be considered zero order in some parameter λ, O_1 is first order, and O_2 is second order. The evolution operator O_0 is chosen to have a simple form $O_0=-\Sigma_q(\partial/\partial h_q)\times[D_q(\partial/\partial h_{-q})+\omega_q h_q]$, where $D_q/\omega_q=\phi_q$. Note that at present ϕ_q and ω_q are not known.

Next, an equation for the two-point function is obtained. The expansion has the form $\phi_q=\phi_q+c_q(\{\phi_p\},\{\omega_p\})$. This reflects the fact that the lowest order in the expansion is exactly the unknown ϕ_q. In the same way an expansion for ω_q is also obtained in the form $\omega_q=\omega_q+d_q(\{\phi_p\},\{\omega_p\})$. Now, the two-point function and the characteristic frequency are thus determined by the two coupled equations

$$c_q(\{\phi_p\},\{\omega_p\})=0,$$

$$d_q(\{\phi_p\},\{\omega_p\})=0. \tag{17}$$

These equations can be solved exactly in the asymptotic limit to yield the required scaling exponents governing the steady-state behavior and the time evolution. Working to second order in the expansion, these equations are just the following two coupled integral equations:

$$D_0 q^{-2\sigma}-\nu_q\phi_q+2\sum_{\ell,m}\frac{M_{q\ell m}M_{q\ell m}\phi_\ell\phi_m}{\omega_q+\omega_\ell+\omega_m} -2\sum_{\ell,m}\frac{M_{q\ell m}M_{\ell mq}\phi_m\phi_q}{\omega_q+\omega_\ell+\omega_m}-2\sum_{\ell,m}\frac{M_{q\ell m}M_{m\ell q}\phi_\ell\phi_q}{\omega_q+\omega_\ell+\omega_m}=0, \tag{18}$$

and

$$\nu_q-\omega_q-2\sum_{\ell,m}M_{q\ell m}\frac{M_{\ell mq}\phi_m+M_{m\ell q}\phi_\ell}{\omega_\ell+\omega_m}=0, \tag{19}$$

where in deriving the last equation I have used the Herring consistency equation [33]. In fact Herring's definition of ω_q is one of many possibilities, each leading to a different consistency equation. But it can be shown, as previously done in Ref. [32], that this does not affect the exponents (universality).

A detailed solution of Eqs. (18) and (19) in the limit of small q's (i.e., large scales) is performed (as in Refs. [16,32]), and yields a very rich family of solutions for any d, ρ, and σ. These solutions are given in Table I. However, before getting into all the details let me focus on the strong-coupling nonpower counting solution obtained by the SCE. The reason for focusing on this solution is that it is interesting to compare it with the solutions obtained by other methods, and with the experimental results. This solution is determined from the combination of the scaling relation $z=(d+4-\Gamma-2\rho)/2$, obtained from Eq. (19), and the transcendental equation $F(\Gamma,z,\rho)=0$, where F is given by

$$F(\Gamma,z,\rho)=-\int d^dt\frac{\vec{t}\cdot(\hat{e}-\vec{t})}{t^z+|\hat{e}-\vec{t}|^z+1}[(\hat{e}\cdot\vec{t})|\hat{e}-\vec{t}|^{-\rho}t^{-\Gamma}+\hat{e}\cdot(\hat{e}-\vec{t})t^{-\rho}|\hat{e}-\vec{t}|^{-\Gamma}]++\int d^dt\frac{[\vec{t}\cdot(\hat{e}-\vec{t})]^2}{t^z+|\hat{e}-\vec{t}|^z+1}t^{-\Gamma}|\hat{e}-\vec{t}|^{-\Gamma}, \tag{20}$$

and $\hat{e}$ is a unit vector in an arbitrary direction.

This solution is valid as long as it satisfies the following conditions: $d<3\Gamma_0(d,\rho)-4+\min\{2\rho-4\sigma,-2\rho\}$ and $d<\Gamma_0(d,\rho)-2|\rho|$, where $\Gamma_0(d,\rho)$ is the numerical solution of the equation $F(\Gamma,z,\rho)=0$—when such a solution exists (how to obtain the conditions above see Ref. [32]).

It turns out that for $d=1$ the equation $F(\Gamma,z(\Gamma),\rho)=0$ is exactly solvable for any ρ, and yields $\Gamma=2+\rho$ and $z=(3-3\rho)/2$ (it can be checked immediately by direct substitution). In this case the three validity conditions read $-1/3<\rho<1$ and $\rho>(4\sigma-1)/5$. By using Eq. (16) I translate the results into $\alpha=(\rho+1)/2$ and $z=(3-3\rho)/2$. It is straightforward to check that this solution reduces to the exact KPZ results of $\alpha=1/2$ and $z=3/2$ in the limit of $\rho=0$. Moreover, this solution recovers the exact solution obtained in Ref. [26], and by doing that clearly performs better than all the other results presented above [Eqs. (5),(7), and (8)].

It should also be mentioned that for $d\geqslant 2$ such an exact solution for $F(\Gamma,z(\Gamma),\rho)=0$ as a closed analytical expression cannot be found, and one has to solve numerically the equation in order to find $\Gamma_0(d,\rho)$ (such results are presented below).

As can be appreciated, the full description of the results given in Table I is quite rich and may look confusing. In order to gain more insight into this seemingly long list of possible phases I concentrate on the case of white noise ($\sigma=0$), and I describe all the possible phases (as a function of ρ) for $d=1,2,3$.

In one dimension, the following scaling exponents are obtained:

$$z=\begin{cases}2, & \rho<-\frac{1}{2}\\ \frac{5-2\rho}{3}, & -\frac{1}{2}<\rho<-\frac{1}{5}\\ \frac{3-3\rho}{2}, & -\frac{1}{5}<\rho<1\\ 2-2\rho, & \rho>1\end{cases}$$

and

$$\Gamma=\begin{cases}2, & \rho<-\frac{1}{2}\\ \frac{5-2\rho}{3}, & -\frac{1}{2}<\rho<-\frac{1}{5}\\ 2+\rho, & -\frac{1}{5}<\rho<1\\ 2, & \rho>1.\end{cases} \tag{21}$$

The dynamic exponent is presented in Fig. 1(a).

It can be seen that the dynamic exponent, obtained using SCE is continuous, has no singularities, and is nonincreasing as a function of ρ. In addition, the intuition that the scaling exponents of the linear theory should appear in the limit of very small ρ's is verified and specified, namely, for $\rho\leqslant-\frac{1}{2}$ the Edwards-Wilkinson (EW) exponents of $z=2$ and $\alpha=1/2$ are obtained.

An important observation regarding the one-dimensional results is that for every region of ρ's, well-defined values of z and Γ are obtained, so that only one phase is possible when ρ is specified, and no phase transition as a function of the dimensionless coupling constant (that is defined by $g^2=\lambda^2D_0/\nu^3$) exists.

In two dimensions, the following scaling exponents are obtained:

$$z=\begin{cases}2, & \rho<0\\ 2-2\rho, & \rho>0\\ \frac{2-\Gamma_0(2,\rho)}{2}+2-\rho, & \rho_2^{\ell}<\rho<\rho_2^{u}\end{cases}$$

and

$$\Gamma=\begin{cases}2, & \rho<0\\ 2, & \rho>0\\ \Gamma_0(2,\rho), & \rho_2^{\ell}<\rho<\rho_2^{u}\end{cases}, \tag{22}$$

where ρ_2^{ℓ} and ρ_2^{u} are the lower and upper bounds on ρ, respectively, for which a solution for the transcendental equation $F(\Gamma,z(\Gamma),\rho)=0$ yields a dynamic exponent that does not exceed $z=2$ and $z=2-2\rho$. To be more specific, $\rho_2^{\ell}=-0.395$ and $\rho_2^{u}=0.305$ [the dynamic exponent is also presented in Fig. 1(b)].

An important observation regarding the two-dimensional result is that for values of ρ within the region $\rho_2^{\ell}<\rho<\rho_2^{u}$ there are two possible solutions (except for $\rho=0$). This can be easily seen in Fig. 1(b)—where two branches appear in that region. This means that a phase transition is possible, as a function of the strength of the dimensionless coupling constant in the theory, between a weak-coupling solution [34] (the upper branch in the figure) and a strong-coupling solution (the lower branch in the figure).

This phenomenon is surprising in view of the known fact that $d=2$ is the lower critical dimension of the local KPZ theory. This means that for dimensions higher than 2 a phase transition becomes possible in the local problem, but in 2D it is only the strong coupling phase. This fact is indeed respected by the results presented above, because for $\rho=0$ (that corresponds to the local theory) such a phase transition is not possible. However, surprisingly if one takes either positive or negative ρ's (even slightly higher or lower than zero) such a phase transition becomes possible. This finding is explained by the fact that the lower critical dimension is lowered in the presence of long-range interactions. The key point is that for positive ρ's ($\rho>0$) the relevant "weak-coupling" phase is a phase with a dynamic exponent of $z=2-2\rho$, and this phase becomes possible for $d>2-\rho$. In addition, for negative ρ's ($\rho<0$), a different weak-coupling

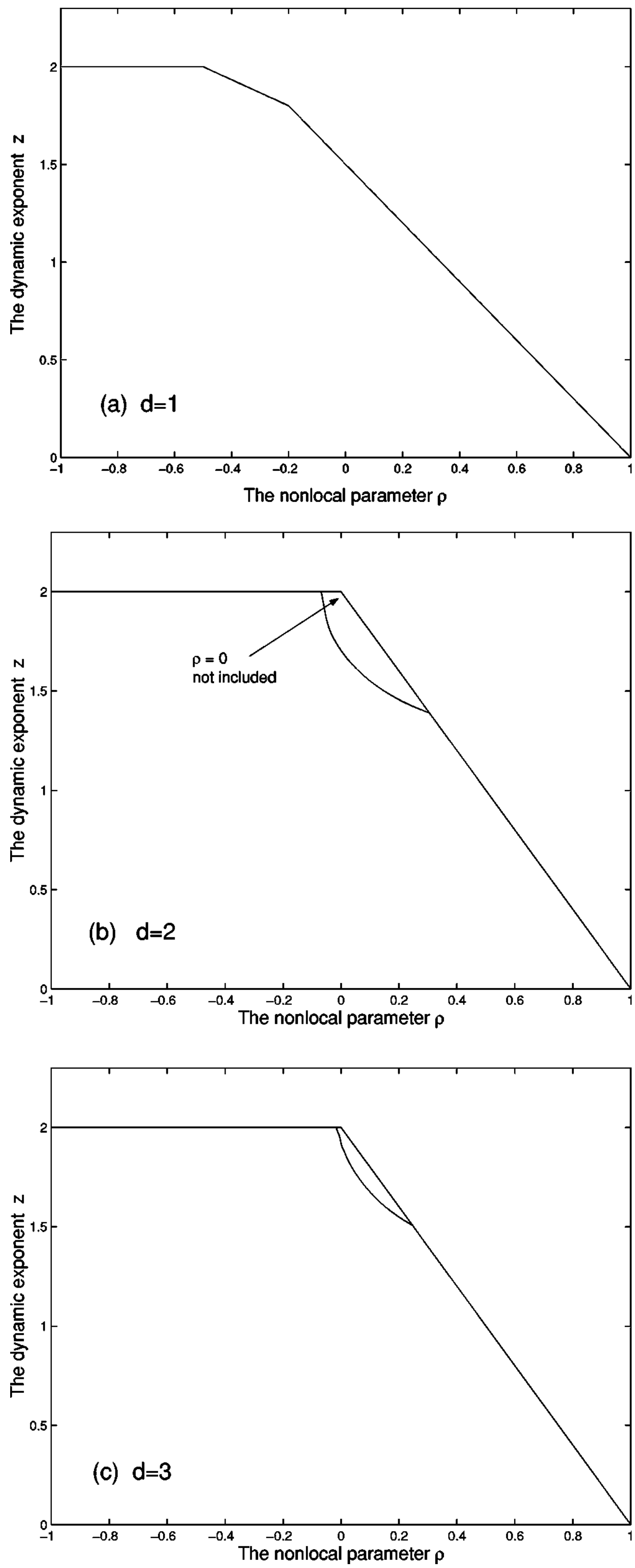

FIG. 1. The values of the dynamic exponent z as a function of the nonlocal parameter ρ, for uncorrelated noise ($\sigma=0$). (a), (b), and (c) are for $d=1,2,$ and $3,$ respectively.

solution is possible—this one with a dynamic exponent of $z=2$, and this phase becomes possible for $d>2+2\rho=2-2|\rho|$. Therefore, once we encounter a nonzero ρ the lower critical dimension is reduced (for two different reasons), and one of the weak-coupling solutions becomes possible. In that case, a phase transition can also take place.

In more than two dimensions, the scaling exponents follow the scheme:

$$z=\begin{cases} 2, & \rho\leqslant 0 \\ 2-2\rho, & \rho>0 \\ \dfrac{d-\Gamma_0(d,\rho)}{2}+2-\rho, & \rho_d^{\ell}<\rho<\rho_d^{u} \end{cases}$$

and

$$\Gamma=\begin{cases} 2, & \rho\leqslant 0 \\ 2, & \rho>0 \\ \Gamma_0(d,\rho), & \rho_d^{\ell}<\rho<\rho_d^{u}, \end{cases} \tag{23}$$

whenever a solution for $F(\Gamma,z(\Gamma),\rho)=0$ exists. Notice a difference from the two-dimensional case in that the weak-coupling exponents are also possible when $\rho=0$. I have performed the detailed calculation for $d=3$, and it is shown in Fig. 1(c) (where $\rho_3^{\ell}=-0.016$ and $\rho_3^{u}=0.246$). It can be seen that no big quantitative difference from the two-dimensional case is present.

IV. DISCUSSION

The results I obtained using the self-consistent expansion (SCE) are obviously different from those obtained by all the other methods. First, the SCE is able to reproduce the exact one-dimensional result, where all the other methods fail. Second, the SCE results are compatible with the expected behavior of the dynamic exponent for $\rho\to-\infty$ (i.e., recovery of the EW exponents) as well as with the expected nonincreasing behavior of the dynamic exponent as a function of ρ. None of the other results mentioned before recovers the EW behavior for large negative ρ's, and only one such result shows a nonincreasing behavior of the dynamic exponent as a function of ρ—namely, the result presented in Ref. [28].

At this point, it might be interesting to return to the experimental result of Refs. [17,18] that were discussed previously, and to discuss them in view of the results derived here. It turns out that the SCE method yields $\rho=0.42$ ($\rho=2\alpha-1$) for the case of $\alpha=0.71$ (at least if the noise is white, or more precisely if the exponent that describes the decay of spatial correlations σ is not large). This result is physically more reasonable than the result $\rho=-0.12$ suggested in Ref. [20]. In addition, since $\rho>0$, it does not require to impose the strict requirement $\lambda_0=0$, which is also problematic physically.

It remains a mystery why all the methods that were mentioned at the beginning of the paper failed to reproduce the exact one-dimensional result, while they all yield the exact result for the local KPZ case ($\rho=0$). The answer to this puzzle lies, in my opinion, in the fact that all those methods fail to notice that the nonlocal nonlinearity generates a nonlocal relaxation term under renormalization. This means that once a nonlocal nonlinearity is introduced into the equation, effectively a certain fractional Laplacian is present as well. This fact can be observed when applying the SCE method, in the ''one-step renormalization stage'' that is part of the solution (see for example Eqs. (6.8)–(6.10) in Ref. [32]). Then, in addition to the q^2-relaxation term a $q^{2-2\rho}$-relaxation term appears, and it obviously dominates the dynamics in the large-scale limit (i.e., $q\rightarrow 0$ limit) when $\rho>0$. I think that the failure to realize this fact prevented the other methods from recovering the exact one-dimensional result.

The last statement can be checked for the methods mentioned above. My prediction is that if one adds by hand the correct relaxational term, namely, $q^{2-2\rho}$, then at least in one dimension the right answer will be obtained for the nonlocal model as well. This idea suggests another important advantage of SCE, since a straightforward application of SCE yields the extra fractional Laplacian, and it does not have to be thought of and added beforehand.

V. CONCLUSIONS

In this paper several results for the nonlocal Kardar-Parisi-Zhang equation have been discussed. It has been shown that those theoretical predictions for the scaling exponents of the NKPZ equation are inconsistent with a recent one-dimensional exact result [26] that is possible for that model. I also claimed that these predictions are not compatible with the expected behavior for the dynamic exponent z as a function of the nonlocal parameter ρ from general considerations presented above.

Then, in order to derive reliable results that are consistent with the exact one-dimensional result an alternative method, namely, the self-consistent expansion, has been used. The results obtained using SCE were also consistent with the expected ρ dependence of the dynamic exponent. In addition, the SCE is able to give sensible predictions for higher dimensions, as well as for any value of ρ and σ. All the possible phases found using SCE are summarized in Table I.

Then I discuss the implication of this calculation to certain experimental results [17,18] that might be well described by a NKPZ model. At the end I also discuss possible reasons for the failure of all the other methods to recover the exact one-dimensional result, and to yield sensible predictions, namely, the generation of a fraction Laplacian term in the equation under renormalization. This observation may trigger future work on the ''fixed'' NKPZ model that includes such a term from the beginning, using methods such as mode coupling. It would then be very interesting to compare between the various results in higher dimensions.

This situation, along with previous results obtained in Refs. [16,31,32], suggests that the SCE method is generally a successful method when dealing with such nonlinear Langevin equations.

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[34] It should be emphasized that the title "weak coupling" here means that such a phase could have been obtained by a linear equation and does not mean that the nonlinear term is not important. The point is that the nonlocal nonlinear term generates a fractional Laplacian (that is more relevant than the Laplacian in the equation) so that the effective linear theory is the Fractal Edwards-Wilkinson equation that gives rise to such "weak-coupling" phases—see discussion for more details.